POLYNOMIALS
3rd LEVEL BILINGUAL SECTION

Polynomials

Introduction

The history of algebra began in ancient Egypt and Babylon, where people learned to solve linear ($ax = b$) and quadratic ($ax^2 + bx = c$) equations, Ancient civilizations wrote out algebraic expressions by using only occasional abbreviations, but by medieval times Islamic mathematicians were able to talk about high powers of unknown values, and work out the basic algebra of polynomials (without using modern symbolism).

This included the ability to multiply, divide, and find square roots of polynomials .

The earliest known use of the equal sign and the signs + for addition, − for subtraction, and the use of a letter for unknown values appears in the 16th century .Later, in 1637, Descartes, in La géometrie, introduced the concept of the graph of a polynomial equation. He popularized the use of letters to denote variables, as we can see in the general formula for a polynomial in one variable. He also introduced the use of superscripts to denote exponents .

We use polynomials in order to represent or model some real life situations. For example,in Physics, when explaining how a ball can be thrown up into the air, and how it comes back down, we use a parabola When figuring out how much Vitamin A you need to make an improvement in your body, we also use a curve (half a parabola).

Contents

- Evaluating polynomials
- Adding,subtracting,multiplying and dividing polynomials
- Factorising polynomials
- Special products
- Algebraic fractions
- Real life problems

A **variable** is a symbol, usually a letter, that we use to represent a number.
Algebraic expressions are combinations of variables, numbers, and at least one operation.
To **evaluate an algebraic expression** ,we replace the variables with numbers.

*Evaluate algebraic expressions
*Add, subtract, multiply,and divide monomials and polynomials.

Example 1. Write an algebraic expression for each verbal expression.

Verbal expression	Algebraic expression
the sum of 15 and k	**15+k**
the difference of 5 and a number	
the product of 12 and x	
28 increased by twice z	
7 decreased by 5 times a number	
9 more than the square of a number	
four fifths the square of b	
three fourths the cube of a number	

Example 2.

a) Evaluate 15 +k if k=3.

We replace k with 3:

$$15+k=15+3=18$$

b) Evaluate $(x-2)^2+3y$ if x=5 and y=-7

We replace x with 5 and y with -7:

$$(5-2)^2+3(-7)=3^2-21=9-21=-12$$

ACTIVITY 1. Write a verbal expression for each algebraic expression.

Verbal expression	Algebraic expression
	4+a
	3-y
	4.x
	15+3s
	3-4t
	$7+\sqrt{x}$
	$\frac{5}{7}\cdot\sqrt{x}$
	$\frac{2}{5}\cdot t^3$

ACTIVITY 2. Evaluate each algebraic expression

$5+\sqrt{x}$	**x=4**	**7**
$3-2z^2$	z=-1	
$-2+a^3$	a=-2	
3x-2y	x=3;y=-2	
$2+3\cdot\sqrt{x}$	x=9	

ACTIVITY 3 . On line lessons (with sound)

Variables

Terms

A **monomial** is a type of algebraic expression. It is the product of a number and one or more variables,so in a monomial, we can only have products and powers with whole exponents:there aren't addition,subtraction,division or powers with negative exponents.

- The **coefficient** is the numerical factor of a monomial
- The **degree of a monomia**l is the sum of the exponents of all its variables
- A monomial without a variable is a **constant**

Example 3. Determine if each expression is a monomial. Explain the reason.

a) $\mathbf{-3x^{-2}z^3}$ is not a monomial because one of the exponents is a negative number.

b) $\mathbf{x^2-2z}$ is not a monomial because there is a difference of two terms

c) $\mathbf{3ab^5}$ is a monomial because there is only product between the variables and the exponents are positive numbers.

d) $\dfrac{5x^3}{y^2}$ ....a monomial because.....................................

Like monomials are monomials that have the same variables and the same exponents.

- 3b and -7b are like monomials
- 3b and $-7b^2$ are not like monomials.

To **add** or **subtract like monomials**,we add or subtract the coefficient of each monomial and write the same variables.

The sum or difference of monomials that are not like is a **polynomial**. Each monomial is a term of the polynomial

Example 4.

- $3a+7a=10a$
- $4x-6x=-2x$
- $5xy^3+2xy^3=7xy^3$
- $7mv^2+2mv^2=$............
- $3x+5x^2$ is not a monomial. It's a polynomial(a binomial).It has two terms.
- $3za-5z^2a+2za^3$ is not a monomial. It's a polynomial(a trinomial).It has three terms.

To **multiply** and **divide** monomials, we multiply (or divide) the coefficients and then, we multiply (or divide) the variables by using the properties of powers.

Properties of powers

- **Product of Powers**
  To multiply two powers that have the same base, we add the exponents.
  $$a^m \cdot a^n = a^{m+n}$$
- **Quotient of Powers**
  To divide two powers that have the same base, we subtract the exponents.
  $$a^m : a^n = a^{m-n}$$
- **Power of a Power**
  To find the power of a power, we multiply the exponents.
  $$(a^m)^n = a^{m.n}$$
- **Power of a Product**
  To find the power of a product, we do the power of each factor and then, we multiply.
  $$(a \cdot b)^n = a^n . b^n$$
- **Power of a Quotient**
  To find the power of a quotient, we do the power of the numerator and the power of the denominator.
  $$(a : b)^n = a^n : b^n$$
- **Zero Exponent**
  Any non zero number raised to the zero power is 1.
  $$a^0 = 1$$

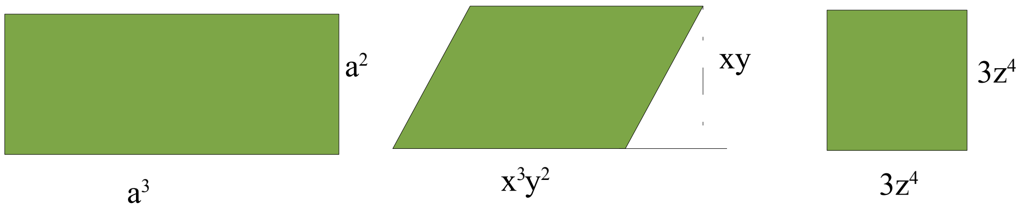
Example 5. Simplify each expression:

$(-2t^3)\cdot(5t^2)$	$-10t^5$
$-5x^2y^4\cdot 3xy$	$-15x^3y^5$
$(-4a^2bc^3)(abc)$	
$(15x^4a^3):(3xa^2)$	
$(-2t^3):(5t^2)$	
$(-4z^2y):(zy)$	
$(x^2y^3z).(xy^2)$	

ACTIVITY 4. Calculate the following operations with monomials. Is the result a monomial in all cases?Why?

a	**b**	**a+b**	**a-b**	**a·b**	**a:b**
$-6t^7$	$-3t^5$				
$-15x^5y^2$	$5x^3y^5$				
$-2abc^6$	abc^2				
$15x^3a^2y$	$-3x^3y$				
$\frac{-3}{2}\cdot t^5$	$\frac{-2}{5}\cdot t$				
z^2y^3	$2xzy$				

ACTIVITY 5. Express the area of each figure as a monomial .



ACTIVITY 6 Lesson with sound

[Multiplication of variables]

A **polynomial** is a monomial or a sum of unlike monomials.

Each monomial is a **term** of the polynomial.

A **trinomial** is a polynomial that has three unlike terms.
Example: $3a^2+3a+5$ is a trinomial

A **binomial** is a polynomial that has two unlike terms.
Example $x+y$ is a binomial

The **degree of a polynomial** is the greatest degree of any term in the polynomial.

The coefficient of the term that has the greatest degree is the **leading coefficient**.

If we change the sign of each term in a polynomial , we get the **opposite** of that polynomial. The opposite of $P(x)$ is $-P(x)$

Evaluate a polynomial is finding out a value of it by replacing the variables with numbers and then, doing all operations.

ACTIVITY 7. Write the variables,the degree,the leading coefficient and the constant term of each polynomial:

P(x)	Variables	Degree	Leading coefficient	Constant term
$-6t^7+5t$				
$-10x^5y^2+2xy$				
ab^2-6abc^6				
$2x^3+x^2-3x+5$				
$\frac{-3}{2}\cdot t^5+\frac{3}{4}\cdot t^2$				
$-xzy$				
y^3-3y				

ACTIVITY 8. Evaluate each polynomial in the previous activity:

P(x)	Values of variables	Value of polynomial
$-6t^7+5t$	t=-2	
$-10x^5y^2+2xy$	x=2;y=-1	
ab^2-6abc^6	a=3;b=1;c=-2	
$2x^3+x^2-3x+5$	x=-1	
$\frac{-3}{2}\cdot t^5+\frac{3}{4}\cdot t^2$	t=2	
$-xzy$	x=5;y=0;z=7	
y^3-3y	$y=\frac{-3}{2}$	

To **add** two polynomials, we group like terms together and then,we add their coefficient .

To **subtract** two polynomials,we add the first one and the opposite of the second one.

Example:

$P(x)=3x^3+3x^2+2x-5$ and $Q(x)=-5x^3-4x+3$

- $P(x)+Q(x)=(3x^3-5x^3)+3x^2+(2x-4x)+(-5+3)=-2x^3+3x^2-2x-2$
- $P(x)-Q(x)=P(x)+(-Q(x))=(3x^3+5x^3)+3x^2+(2x+4x)+(-5-3)=8x^3+3x^2-8$

$-Q(x)=5x^3+4x-3$

To multiply two polynomials ,we multiply each monomial of the first polynomial by each monomial in the second polynomial. Then, we add the like terms.

Example:

$P(x)=3x^3+3x^2+2x-5$ and $Q(x)=-5x^3-4x+3$

$P(x)\cdot Q(x)=(3x^3+3x^2+2x-5)\cdot(-5x^3-4x+3)$

$$
\begin{array}{rr}
 & 3x^3+3x^2+2x-5 \\
 & -5x^3-4x+3 \\
\hline
 & 9x^3+9x^2+6x-15 \\
 & -12x^4-12x^3-8x^2+20x \\
 & -15x^6-15x^5-10x^4+25x^3 \\
\hline
P(x)\cdot Q(x)= & -15x^6-15x^5-22x^4+22x^3+x^2+26x-15
\end{array}
$$

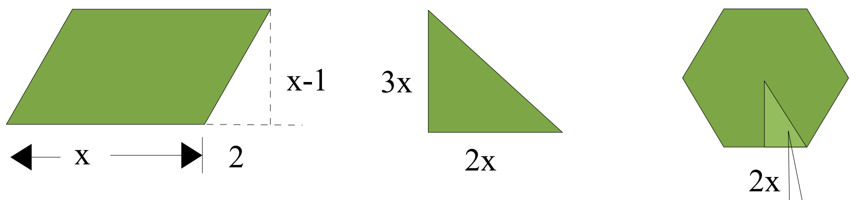
ACTIVITY 9. Find the sum, difference and product of the following polynomials

P(x)	Q(x)	P(x)+Q(x)	P(x)-Q(x)	P(x)·Q(x)
$-6t^7+5t$	$-3t^7+t^2-t$			
$-10x^5y^2+2xy$	$3x^5y^2-2xy$			
ab^2-6abc^6	abc^2-3abc^3			
$2x^3+x^2-3x+5$	$-3x^3-5x+2x-1$			
$\frac{-3}{2}\cdot t^5+\frac{3}{4}\cdot t^2$	$\frac{-2}{5}\cdot t^2+\frac{1}{5}\cdot t^5$			
-xzy	-xzy			
y^3-3y	$-2y^3+5y$			

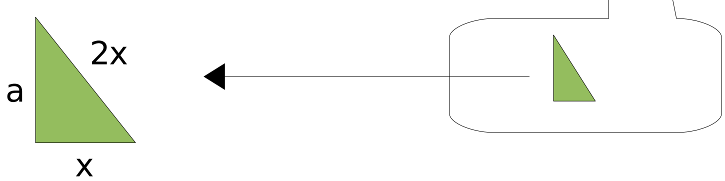
ACTIVITY 10. Crossword

[Adding polynomials]

ACTIVITY 11. Express the perimeter and the area of each figure as a polynomial



- Remember:
  - **Regular polygon area**= $\frac{(P\cdot a)}{2}$,where P is the perimeter and a is the apothem.
  - The radius and the side in an hexagon are equal
  - To calculate the apothem,a, you have to use Pythagoras Theorem in this triangle:



When you **divide** a polynomial D(x) by another polynomial,d(x),you get two polynomials Q(x) and R(x) that verify the following expression:

D(x)=Q(x).d(x)+R(x)

Degree of R(x)<Degree of d(x)

D(x) is the dividend
d(x) is the divisor
R(x) is the remainder

There are two methods to calculate the division of two polynomials:

- **Long division**. You can use this method to calculate any division.
- **Synthetic division (Ruffini Rule).** You can only use this method to divide a polynomial by a binomial x+a.

Example 6.
Find $2x^3-x^2+5x-4 : x+3$ by using long division and synthetic division.

$D(x)=2x^3-x^2+5x-4$ is the **dividend** and $d(x)=x+3$ is the **divisor**

- **Long division**

$2x^3-x^2+5x-4$	$x+3$ $2x^2$	Divide the first term of the dividend, $2x^3$,by the first term of the divisor,x.
$2x^3 -x^2+5x-4$ $-2x^3-6x^2$ $-7x^2+5x-4$	$x+3$ $2x^2$	Multiply $2x^2$ and x+3 and subtract
$2x^3 -x^2+5x -4$ $-2x^3-6x^2$ $-7x^2+5x -4$ $7x^2+21x$ $26x-4$ $-26x-78$ -82	$x+3$ $2x^2 -7x+26$	• Now,divide $-7x^2$ by x that is -7x. • Multiply -7x by x+3 y do the difference with $-7x^2+5x-4$ • Divide 26x by x, multiply by x+3,then subtract
The **quotient ,Q(x)**, is $2x^2 -7x+26$ The **remainder,R(x),** is -82 We can write **D(x)=d(x)·Q(x)+R(x)**		

The previous process is over when the degree of the remainder is less than the divisor's one.

- **Synthetic division**

  $2x^3-x^2+5x-4 : x+3$

$2x^3 \; \boxed{-} \; x^2 + 5\,x - 4$ 2 -1 5 -4	Firstly,we arrange the dividend in descending order. Then, we just write the coefficients of the term of the dividend
2 -1 5 -4 -3 -6 2 -7	Write the opposite of the constant of the divisor ,-3 ,to the left. Bring down the first coefficient,2 . Multiply the constant -3 by 2 , write the product under the next coefficient and add.
2 -1 5 -4 -3 -6 21 2 -7 26	Multiply -3 by -7,write the product under the next coefficient and add.
2 -1 5 -4 -3 -6 21 -78 2 -7 26 -82	Multiply this sum by -3. Write the product under the next coefficient and add.
$2x^3 -7x^2 +26$ **Quotient** -82 **Remainder**	The numbers along the bottom are the coefficients of the quotient. And the last number is the remainder
2 -1 5 -4 -3 -6 21 -78 2 -7 26 -82	This is all the process in an only picture.

ACTIVITY 12. Find the quotient and the remainder of the following divisions by using synthetic division when it is possible:

- **a)** $x^3-3x^2+x-1:x^2-1$
- **b)** $z^4-3z+2:z$
- **c)** $x^5-3x^4+2x-6:x^4+x$
- **d)** $x^3-3x^2+5:x-3$

Factoring a polynomial is to write it as a product of factors.
For example, we can write $\mathbf{2x^2+2x}$ as $\mathbf{2x(x+1)}$ where $\mathbf{2x}$ and $\mathbf{x+1}$ are the **factors**.
There are several techniques to factor a polynomial. In this unit we are going to use:

- Factoring out the common factors of the polynomial
- Recognizing the difference of two squares, and recognizing perfect squares of binomials.(special products)

Factoring out the common factors of the polynomial

Firstly we write each term as a product .Secondly, we identify the greatest common factor of all the terms. Then we apply the distributive property(the reverse of this property).

Distributive property: a(b+c)=a.b+a.c

Example7 : Factor out the common factor of the polynomial

$$6x^4-3x^3+12x^2$$

Write each term as a product :

$$2.\boxed{3}.\boxed{x.x}.x.x-\boxed{3}x.\boxed{x.x}+2.2.\boxed{3}.\boxed{x.x}$$

Identify the greatest common factor :

$$3.x.x=3.x^2$$

Applying the reverse of the distributive property:

$$2.\boxed{3.x.x}.x.x-\boxed{3x.x}.x+2.2.\boxed{3.x.x}=3x^2.(2x^2-x+4)$$

$$6x^4-3x^3+12x^2=3x^2.(2x^2-x+4)$$

ACTIVITY 13. Interactive activity

Common factors and factorising

ACTIVITY 14. Crossword

Factor out the GCF

ACTIVITY 15. Fill in the blanks

a) $x^2-3x=x(\square - \square)$

b) $-20+15z^3 = \square.(4-3\square)$

c) $9x^3-3x^2 = 3.\square(x-\square)$

d) $12x^4.y^3-18x^3.y^4= \square x^{\square}.y^{\square}.(2x-3y)$

ACTIVITY 16. Factor out the common factor from each polynomial:

a) y^2-6y
b) $-12+16x^5$
c) $5z^3-10z^4$
d) $30x^5.y^2-15x^2.y^5$

Special products

- **Square of a Sum**
  The square of a + b is the square of a plus twice the product of a and b plus the square of b.
  $$(a+b)^2 = a^2+2ab+b^2$$
- **Square of a Difference**
  The square of a - b is the square of a minus twice the product of a and b plus the square of b.
  $$(a-b)^2 = a^2-2ab+b^2$$
- **Product of a Sum and a Difference**
  The product of a + b and a - b is the square of a minus the square of b.
  $$(a+b).(a-b) = a^2-b^2$$

Example 8: Find the following squares of sums or differences:

a)$(x+5)^2=x^2+2.x.5+25=x^2+10x+$........
b)$(x-3)^2=x^2-$........$+9$
c)$(5x-2)^2=(5x)^2-2.5x.2+2^2=25x^2-20x+4$
d)$(7x-3)=($.......$)^2-2$.....$3+3^3=$..........$-42X+$..........

Example 9 :Find the product of a sum and a difference

a)$(x-2)(x+2)=x^2-2^2=x^2-4$
b)$(3x-5).(3x+5)=(3x)^2-5^2=9x^2-25$
c)$(7x+2)(7x-2)=($......$)^2-$........$^2=49x^2-4$

ACTIVITY 17. Match each expression in the first column with its equivalent in the second one

$(2a^2-3b)^2$
$(a^2+b)(a^2-b)$
$(2a+b)^2$
$(3a+2b)(3a-2b)$

$9a^2-4b^2$
$4a^2+4ab+b^2$
$4a^4-12a^2b+9b^2$
a^4-b^2

Complete the table so that each expression in the first column is equal to the expression in the second one

$10a^2b^5-5a^5b^2$	5......$(2b^3-$........)
$3x^2-12x$	3x(........-.........)
(........$+zy)^2$	$25y^4+$..........$+z^2y^2$
(........+...........$)^2$	$4x^2+$...........$+9$
(......-2)(.......+2)	x^4 -........
(.........-..........$)^2$	$25a^2$-.........$+b^2$

ACTIVITY 18. Interactive activity

[Special products]

An **algebraic fraction** is the quotient of two polynomials.
To simplify an algebraic expression, firstly, we factor the numerator and the denominator by using distributive property (common factor) or special products.
Then we divide each one by a common factor (usually the GCF)

Example 10

$$\frac{7x^3z}{3z^2x} = \frac{7x^2\cdot\cancel{x}\cdot\cancel{z}}{3z\cdot\cancel{z}\cdot\cancel{x}} = \frac{7x^2}{3z}$$

We can cancel x and z because they are common factors of the numerator and the denominator

*Factoring polynomials by using common factors or special products
*Simplifying rational expressions by factoring both numerator and denominator

Example 11

$$\frac{x^2-5x}{3x} = \frac{\cancel{x}(x-5)}{3\cancel{x}} = \frac{x-5}{3}$$

We factorise the numerator by using that x is a common factor of x^2 and 5x

We can cancel x because it is a common factor of the numerator and the denominator

Example 12

$$\frac{9x^2-12x+4}{3x-2}=\frac{(3x-2)^2}{3x-2}=3x-2$$

We factorise the numerator by using a special product(the square of a sum)

We can cancel 3x-2 because it is a common factor of the numerator and the denominator

Example 13

$$\frac{25x^2-4}{15x-6}=\frac{(5x+2)(5x-2)}{3(5x-2)}=\frac{5x+2}{3}$$

We factorise the numerator by using a special product(the difference of two squares)

We can cancel 5x-2 because it is a common factor of the numerator and the denominator

ACTIVITY 19

$$\frac{7a^3b}{2a^2b^2} = \frac{7a\square\square b}{2\square\square\square\square} = \frac{7a}{2b}$$

$$\frac{4x^2-12x+9}{2x-3} = \frac{\left(\square-\square\right)^2}{2x-3} = 2x-3$$

$$\frac{9x^2-1}{6x-2} = \frac{\left(\square+1\right)\left(\square-1\right)}{2\left(\square-\square\right)} = \frac{\square}{\square}$$

$$\frac{x^2-6x+9}{x(x-3)} = \frac{\left(\square-\square\right)^2}{x(x-3)} = \frac{\square}{\square}$$

$$\frac{(2x+5)(x+3)}{4x^2-25} = \frac{(2x+5)(x+3)}{\left(\square+\square\right)\left(\square-\square\right)}$$

ACTIVITY 20. On-line lesson (with sound)

Algebraic fractions

ACTIVITY 21. Revise your vocabulary
Choose a word in the box and fill the blanks below:

leading coefficient, degree algebraic expressions, like , monomial polynomial , constants

- ______ are combinations of variables, numbers, and at least one operation.
- ______ is the product of a number and one or more variables.
- ______ are the monomials that are only numbers.
- ______ monomials are monomials that have the same variables and the same exponents.
- A sum of unlike monomials is a ______
- The greatest degree of any term in a polynomial is the ______ of the polynomial
- The coefficient of the term that has the greatest degree is the ______

ACTIVITY 22. Revise your vocabulary
Write the sentences in the correct order:

- by replacing /is finding out /Evaluate a polynomial /the variables with numbers/a value of it
- and then /we group /To add,two polynomials/like terms together /we add their coefficient .
- two powers /To divide /we subtract the exponents./that have the same base,
- we multiply the variables /To multiply monomials,/by using the properties of powers./ we multiply the coefficients and then
- two powers/To multiply /,we add the exponents/that have the same base .
- is the quotient /An algebraic fraction /of two polynomials.
- is to write it /Factoring /as a product of factors/a polynomial

ACTIVITY 23.

Anna is doing an experiment in the laboratory. She finds that there are $3x^7$ particles in one container.
There are $4x^7+2$ particles in another one.
How many particles are there if she combines the two containers?

*Real life situations

ACTIVITY 24.

There are about $5 \cdot 10^6$ red blood cells in one millilitre of blood. A certain blood sample contains $8 \cdot 10^{32}$ red blood cells. About how many millilitres of blood are there in the sample?

ACTIVITY 25.

The polynomial $x^3-70x^2+1500x-10,800$ represents the profit that a company makes on selling an item at a price x. A second item sold at the same price brings in a profit of $x^3-30x^2+450x-5000$.Write a polynomial that expresses the total profit from the sale of both items.

ACTIVITY 26.

A rock falls from the top of a mountain 82 meters above the ground. Use the formula $e= 5t^2$, where e is the distance the rock falls and t is the time in seconds of the fall, to determine approximately how long it takes for the rock to land on the ground.

ACTIVITY 27.

You are planning a rectangular garden. Its length is twice its width x. You want a walkway 1,5 m wide around the garden.

a) Write an expression for the area of the garden and walkway.
b) Write an expression for the area of the walk only

You have enough gravel to cover 90 m^2 and you want to use it all on the walk. How big should you make the garden?

ACTIVITY 28.-Card Sort Activity

It can be used as a Worksheet or as Game
If is used as a game,students will work in groups of three or four to match each each special product with its corresponding expression. Each group must justify their responses.

*Game

$(x+3)^2$	$(x-y)$	$(2x-3)(2x+3)$
$(y-1)^2$	$(3y-5)(3y+5)$	$(3x+4y)^2$
$(y-3x)^2$	$(y-2)(y+2)$	$(x^2+1)(x^2-1)$

x^2+6x+9	x^2-2x+y^2	$4x^2-9$
y^2-2y+1	$9y^2-25$	$9x^2+24xy+16y^2$
$y^2-6yx+x^2$	y^2-4	x^4-1

Revision and Revision Test

Evaluating algebraic expressions	▪ Test FINDING VALUES
Adding and multiplying polynomials	▪ Test ADDING POLYNOMIALS ▪ Test REMOVING BRACKETS ▪ Test MULTIPLYING POLYNOMIALS
Dividing polynomials	▪ Test SYNTHETIC DIVISION
Factorising and simplifying	▪ Test SIMPLIFYING
Algebraic fractions	▪ Worksheet SIMPLIFYING ALGEBRAIC FRACTIONS