# EXPONENTIAL AND LOGARITHMIC FUNCTIONS
# 4th LEVEL BILINGUAL SECTION
# SCHOOL YEAR 2009/2010

# Exponential and logarithmic functions

## Introduction

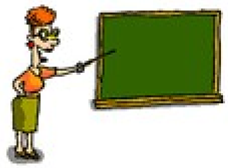
Logarithms were developed to facilitate computations in Astronomy and Trigonometry. Some mathematicians of the 16$^{th}$ century tried to coordinate arithmetical and geometrical progressions in order to work easily with the complicated trigonometrical tables.

Napier (1550-1617), Scottish mathematician and Burgi (1552-1632), Swiss mathematician worked independently in this field. However, Napier is known as the inventor of logarithms because he published his work first.

The idea was to construct two sequences of numbers so related that when one increases in arithmetical progression,the other decreases in a geometrical one. Napier and Briggs decided to set the logarithm of 1 as 0 so that the tables of logarithms could be easier to use.

Nowadays, we explain and study logarithms through exponentials but historically discoveries didn't follow this order. Actually,the concept of an exponential function dates from the later part of the 17$^{th}$ century. Euler(1707,1783) first used the letter e for the base of natural logarithms based on the exponential function $y=e^x$.
The use of logarithm as a computing technique has disappeared today but as we are going to see in some examples and activities,logarithmic functions are now applied to model many situations in Geology,Economy,Chemistry,Biology,Geography.

## Contents



# Exponential functions. Properties

An **exponential function** is an equation of the form

$$f(x)=a^x \quad (\text{or } y=a^x)$$

where $a>0$ and $a\neq 1$

**Properties**

- The graph of $y=a^x$ ( $a>0$ y $a\neq 1$) never has a X-intercept.

  The **Y-intercept** is always (0,1)

- If $a>1$,the function $y=a^x$ is **increasing** (exponential growth).

  If $0<a<1$, the function $y=a^x$ is **decreasing** (exponential decay)

- The **domain** of $f(x)=a^x$ ( $a>0$ and $a\neq 1$) is all real numbers because we can evaluate f at any real number(rational or irrational)

- The **range** of f is the positive real numbers because $a^x>0$ for all values of x

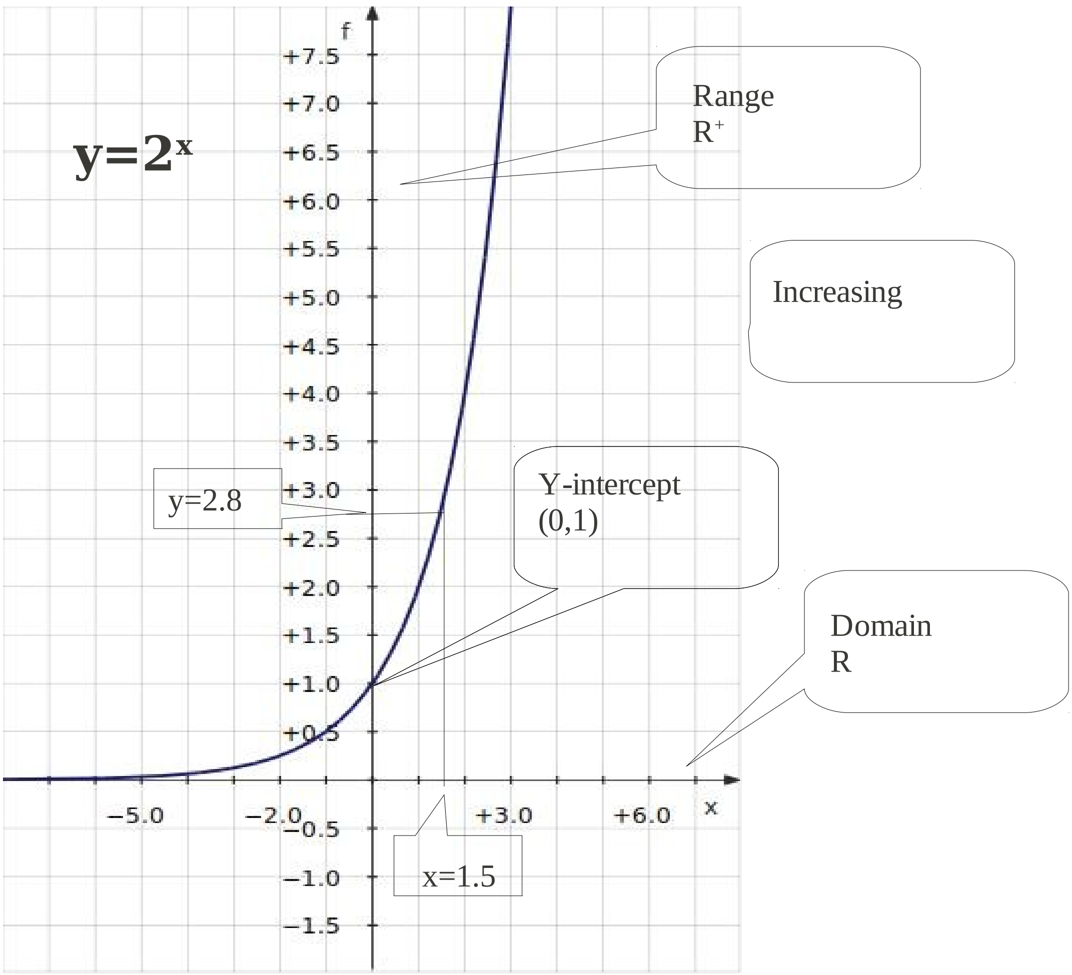
These properties can change when we apply a translation to the function $f(x)=a^x$

**Note:**
In exponential functions .the independent variable is the exponent whereas ,in polynomial functions,the independent variable is the base and the exponent is constant

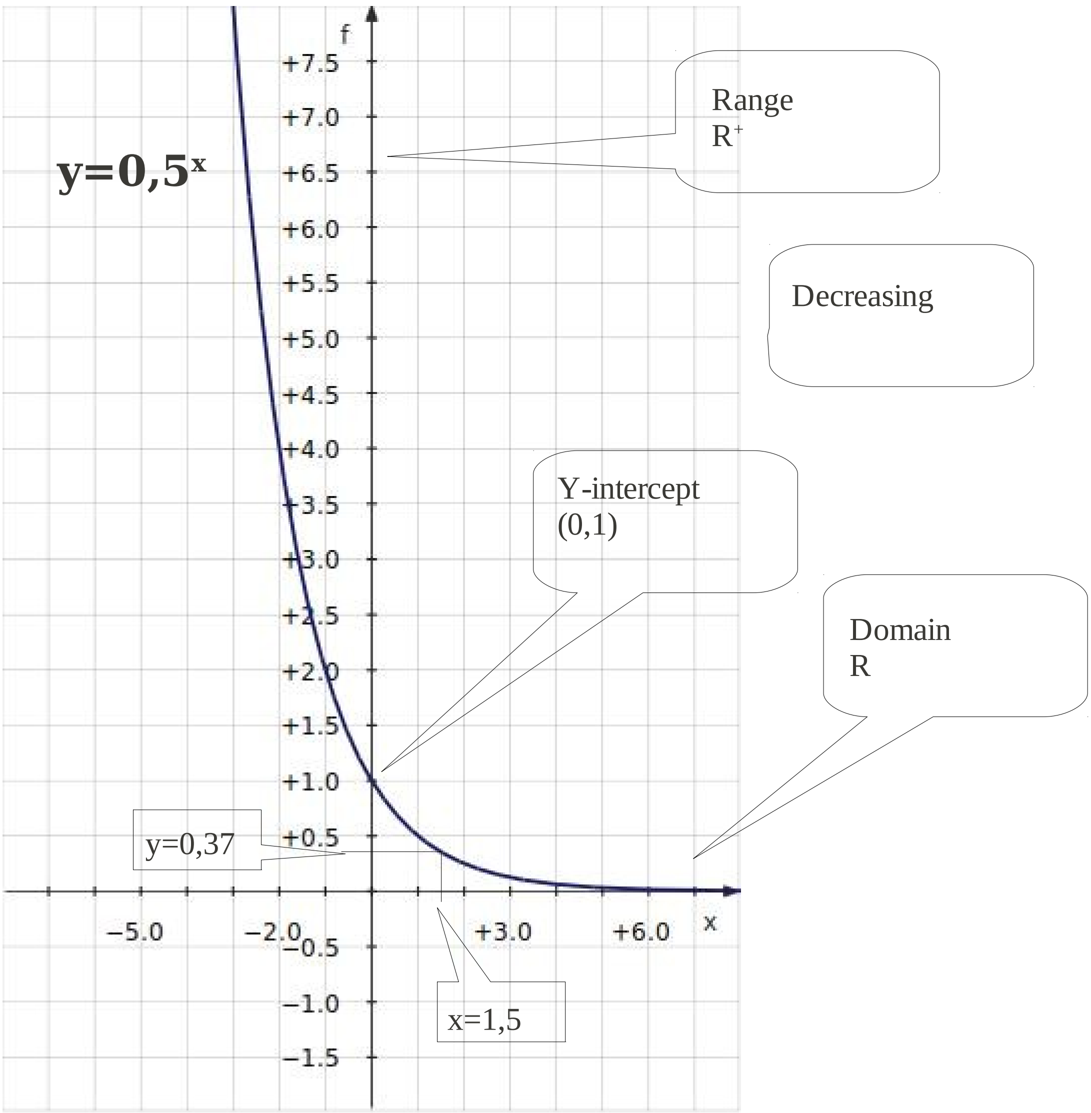
When studying exponential functions it can be useful to revise the **properties of powers**

**Example 1.** Make a values table and graph the functions $y=2^x$ and $y=\left(\frac{1}{2}\right)^x$

| x | $y=2^x$ |
|---|---|
| -3 | 0,125 |
| -2 | 0,25 |
| -1 | 0,5 |
| 0 | 1 |
| 1 | 2 |
| 2 | 4 |
| 3 | 8 |



| x | y= $\left(\frac{1}{2}\right)^x$ |
|---|---|
| -3 | 8 |
| -2 | 4 |
| -1 | 2 |
| 0 | 1 |
| 1 | 0.5 |
| 2 | 0.25 |
| 3 | 0,125 |



**Example 2.**

a) Use the graphs above to determine the approximate value of $2^{1,5}$ and $\left(\frac{1}{2}\right)^{1.5}$

*If we look at the graph of y= $2^x$ ,we observe that the value of y is about 2.8 when x=1.5. Therefore, we can say that $2^{1,5}$ is approximately 2.8.*

*If we look at the graph of y= $\left(\frac{1}{2}\right)^x$ ,we observe that the value of y is about ………… when x=1.5. Therefore, we can say that $\left(\frac{1}{2}\right)^{1.5}$ is approximately …………..*

b)

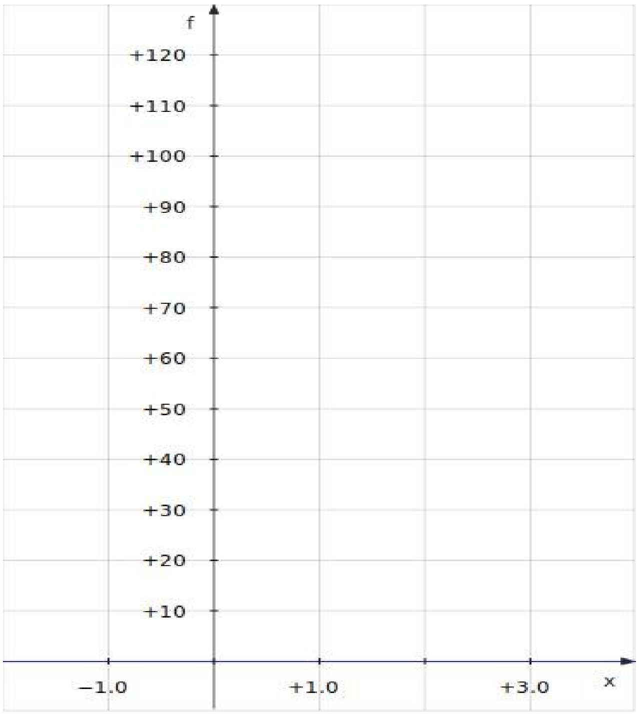
i. Evaluate $y=2^x$ at x=1,5 using the $x^y$ bottom on your calculator .Round to the nearest ten-thousandth.

*$2^{1,5}\approx 2$ $x^y$ = 2.828427124746. Rounding to the nearest ten-thousandth,we get $2^{1.5}\approx 2.8284$*

ii. Evaluate y= $\left(\frac{1}{2}\right)^x$ at x=1,5 using the $x^y$ bottom on your calculator .Round to the nearest ten-thousandth.

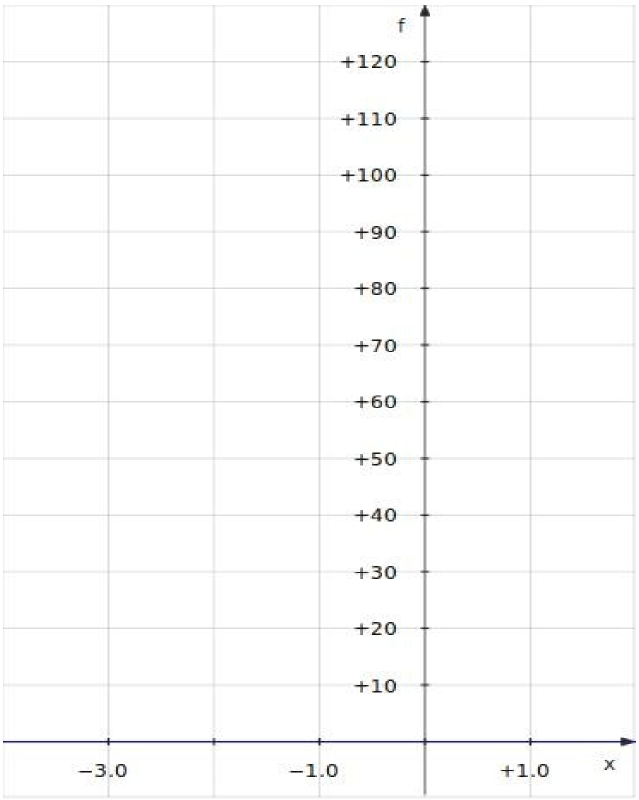
$\left(\frac{1}{2}\right)^{1,5} \approx \frac{1}{2}$ $x^y$ *1,5* = ………………………………… *Rounding to the nearest ten-thousand ,we get* $\left(\frac{1}{2}\right)^{1,5} \approx$ ………………………………

**ACTIVITY 1.** Make a values table and graph the functions $y=5^x$ and $y=\left(\frac{1}{5}\right)^x$. State the Y-intercept, the domain and the range.




| x | $y=5^x$ |
|---|---|
| -2 | |
| -1 | |
| 0 | |
| 1 | |
| 2 | |
| 3 | |

| $y=5^x$ | |
|---|---|
| **Domain** | |
| **Range** | |
| **Y-intercept** | |




| x | $y=\left(\frac{1}{5}\right)^x$ |
|---|---|
| -3 | |
| -2 | |
| -1 | |
| 0 | |
| 1 | |
| 2 | |

| $y=\left(\frac{1}{5}\right)^x$ | |
|---|---|
| **Domain** | |
| **Range** | |
| **Y-intercept** | |

**ACTIVITY 2.**Use the graphs in **ACTIVITY 1** to find the approximate value of $5^{2,5}$ and $\left(\frac{1}{5}\right)^{-2.5}$

Then use the calculator and compare the results.

| | **Using the graph** | **Using the calculator** |
|---|---|---|
| $5^{2,5}$ | | |
| $\left(\frac{1}{5}\right)^{-2.5}$ | | |

Write the approximation to the nearest hundred-thousandth

**ACTIVITY 3 .** On line activities

**Exponential growth of a bacteria colony**

**Growth of a bacteria colony-Video**

# Translations and stretching

An **horizontal translation** of the exponential function $y=a^x$ (a>0) is any function of the form $y=a^{x+k}$ where k is a constant.

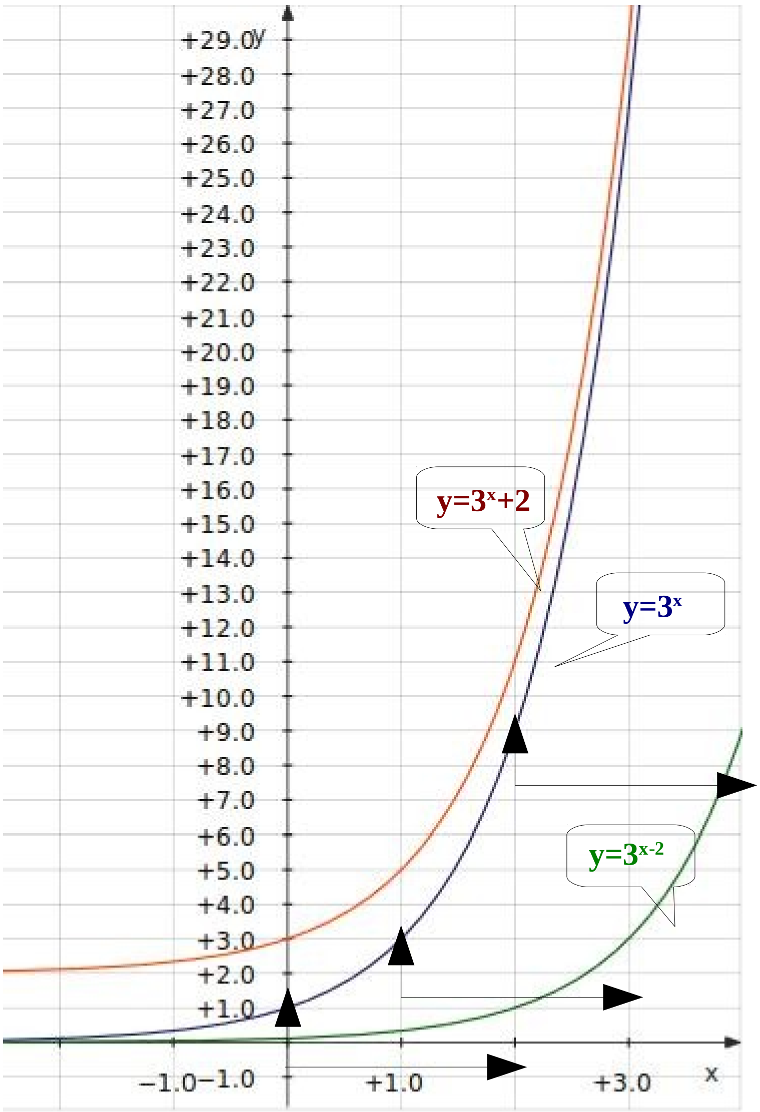
- If **k>0,**we shift the graph of $y=a^x$ k units to the **left** in order to get the graph of $y=a^{x+k}$
- If **k<0**,we shift the graph of $y=a^x$ k units to the **right** in order to get the graph of $y=a^{x+k}$

A **vertical translation** of the exponential function $y=a^x$ (a>0) is any function of the form $y=a^x+k$ where k is a constant.

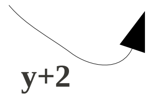
- If **k>0**,we shift up k units the graph of $y=a^x$ in order to get the graph of $y=a^x+k$
- If **k<0**,we shift down k units the graph of $y=a^x$ in order to get the graph of $y=a^x+k$

**Note:**
In horizontal translations, we add or subtract a constant in the exponent
In vertical translations,we add or subtract a constant in the exponential term.

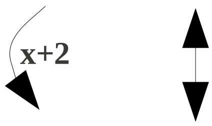
**Example 3.** Graph the functions $y= 3^x$ , $y=3^{x-2}$ and $y= 3^x+2$ on the same coordinate plane.




| x | $y=3^x$ | $y=3^x+2$ |
|---|---|---|
| 0 | 1 | 3 |
| 1 | 3 | 5 |
| 2 | 9 | 11 |




| x | $y=3^x$ |
|---|---|
| 0 | 1 |
| 1 | 3 |
| 2 | 9 |




| x | $y=3^{x-2}$ |
|---|---|
| 2 | 1 |
| 3 | 3 |
| 4 | 9 |

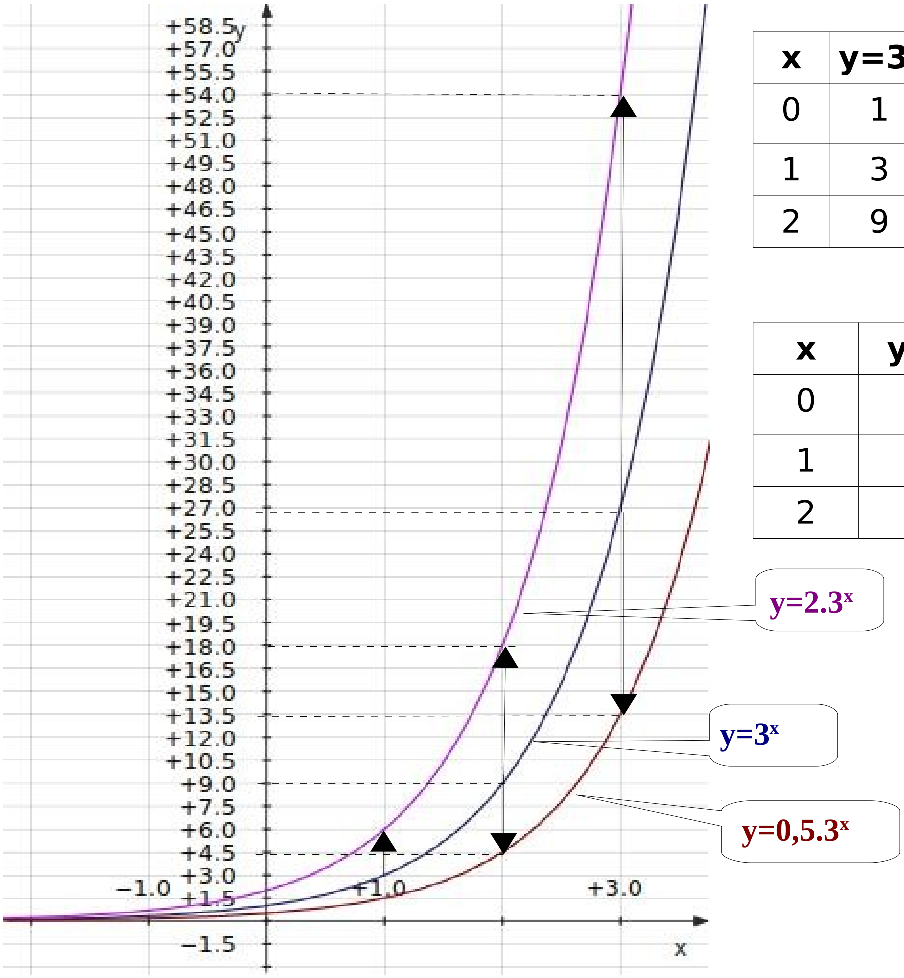
**Vertical Stretching** : $\mathbf{y=k.a^x}$ , k is a constant

- If **0<k<1**,we can obtain the graph of $y=ka^x$ by shrinking the graph of $y=a^x$ vertically,multiplying each value of $a^x$ by k.
- If **k>1**,we can obtain the graph of $y=ka^x$ by stretching the graph of $y=a^x$ vertically,multiplying each value of $a^x$ by k.

**Horizontal Stretching** : $\mathbf{y=a^{kx}}$ , k is a constant .

- If **0<k<1**,we can obtain the graph of $\mathbf{y=a^{kx}}$ by stretching the graph of $y=a^x$ horizontally ,multiplying each x-value of $y=a^x$ by 1/k.
- If **k>1**,we can obtain the graph of $\mathbf{y=a^{kx}}$ by shrinking the graph of $y=a^x$ horizontally ,multiplying each x-value of $y=a^x$ by 1/k.

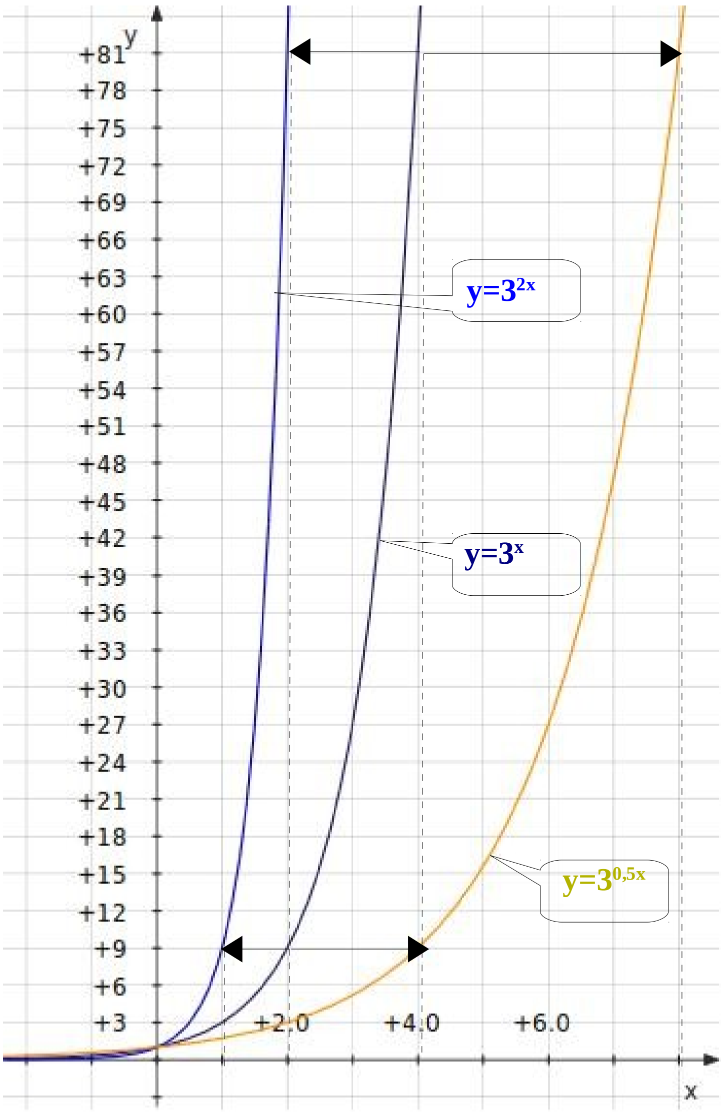
**Example 4.**Graph the functions $y= 3^x$ , $y=2.3^x$ and $y=1/2.3^x$ on the same coordinate plane.




| x | $y=3^x$ | $y=2.3^x$ |
|---|---|---|
| 0 | 1 | 3 |
| 1 | 3 | 6 |
| 2 | 9 | 18 |

| x | $y=3^x$ | $y=1/2.3^x$ |
|---|---|---|
| 0 | 1 | 0,5 |
| 1 | 3 | 1,5 |
| 2 | 9 | 4,5 |

**Example 5.** Graph the function $y=3^{2x}$ and $y=3^{0,5x}$ using the table of the function $y=3^x$




| x | $y=3^x$ |
|---|---|
| 0 | 1 |
| 2 | 9 |
| 4 | 81 |

x/2

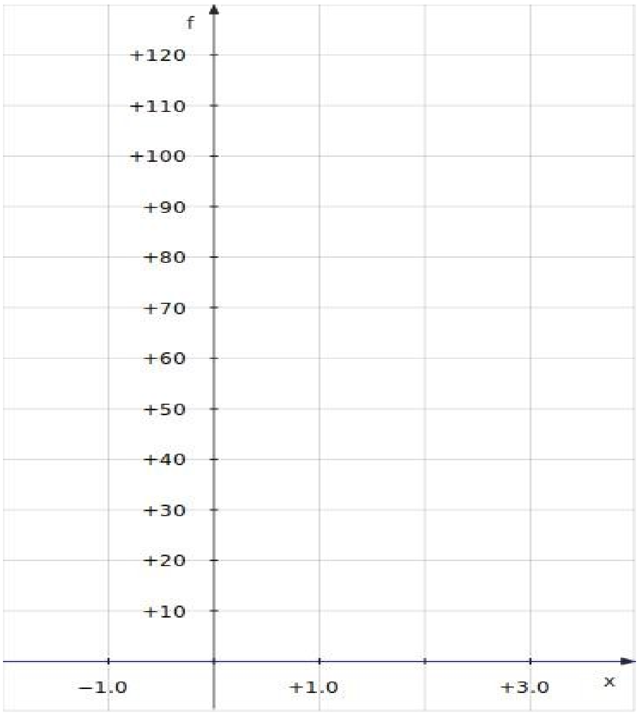
| x | $y=3^{2x}$ |
|---|---|
| 0 | 1 |
| 1 | 9 |
| 2 | 81 |

| x | $y=3^x$ |
|---|---|
| 0 | 1 |
| 2 | 9 |
| 4 | 81 |

2.x

| x | $y=3^{0,5x}$ |
|---|---|
| 0 | 1 |
| 4 | 9 |
| 8 | 81 |

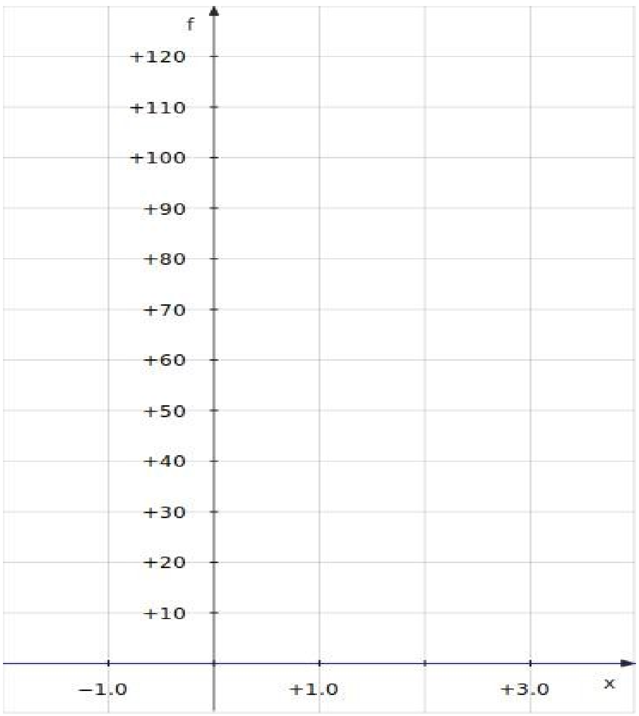
**ACTIVITY 4.** Graph the functions $y=5^{x+1}$ and $y=5^{x}-10$ using the function $y=5^{x}$




| x | $y=5^x$ | $y=5^x-10$ |
|---|---|---|
| -1 | 0,2 | |
| 0 | 1 | |
| 1 | 5 | |
| 2 | 25 | |
| 3 | 125 | |

| x | $y=5^{x+1}$ |
|---|---|
| | 0,2 |
| | 1 |
| | 5 |
| | 25 |
| | 125 |

**ACTIVITY 5.** Graph the functions $y=4.5^{x}$ and $y=5^{2x}$ using the function $y=5^{x}$

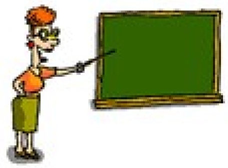



| x | $y=5^x$ | $y=4.5^x$ |
|---|---|---|
| -1 | 0,2 | |
| 0 | 1 | |
| 1 | 5 | |
| 2 | 25 | |

| x | $y=5^{2x}$ |
|---|---|
| | 0,2 |
| | 1 |
| | 5 |
| | 25 |
| | 125 |

**ACTIVITY 6.** Interactive activity:manipulating functions by using an applet
Function Data

**Note:**
We often use the irrational number e (approximately 2,718281....) as the base of an exponential function ($y=e^x$) to model many natural and social phenomena



# Logarithmic functions

**Example 6.**
A certain bacteria splits into two bacteria every hour,so the number of bacteria in a culture doubles every hour. The table below shows the number of bacteria in the culture after 1,2,3,4,...hours.

| t | N |
|---|---|
| 1 | 2 |
| 2 | 4 |
| 3 | 8 |
| 4 | 16 |
| 5 | ... |

a) The number of bacteria after 5 hours will be $2^5$
The number of bacteria after 6 hours will be $2^6$
The number of bacteria after 7 hours will be $2^7$
..........................................................................................................

The number,N, of bacteria after t hours will be $2^t$

b) The exponential equation that relates the time to the number of bacteria is:

$$N=2^t$$

c) How many bacteria will there be after 15 hours?

$$N=2^{15}=32768 \text{ bacteria}$$

d) In how many hours will the culture have 92000 bacteria?
Now we want to calculate t instead of N,so we can write :

$$92000=2^t$$

We need to find the exponent to which we have to raise the base 2 to obtain 92000.

This is called **logarithm of 92000 with base 2** which is denoted **$\log_2 92000$.**

We use logarithm properties (the change of base formula) and the calculator to find this value. (You are going to calculate **$\log_2 92000$** in **ACTIVITY 7**)

$92000=2^t \Longleftrightarrow$ t is the exponent of 2 that produces 92000 $\Longleftrightarrow t=\log_2 92000$

e) We can write an exponential equation as a logarithmic equation and a logarithmic equation as an exponential equation.

| Exponential form | Logarithmic form |
|---|---|
| $2^6=64$ | $\log_2 64=6$ |
| $\left(\frac{1}{5}\right)^{-2}=25$ | $\log_{1/5}25=-2$ |
| $10^3=1000$ | $\log_{10}1000=3$ |
| $e^2\approx 7{,}389$ | $\log_e 7{,}389\approx 2$ |
| $5^{-1}=0{,}2$ | log $_5$........=...... |
| .......... | $\ln 7\approx 1{,}946$ |

When the base is 10,we never write $\log_{10}1000=3$ but $\log 1000=3$ This kind of logarithm is called decimal(or common) logarithm

When the base is e,we never write $\log_e 7{,}389\approx 2$ but $\ln 7{,}389\approx 2$ This kind of logarithm is called natural logarithm*

*or neperian (from Napier)logarithm

If a and x are positive numbers, $a \neq 1$, the logarithm of x with base a is the exponent y to which we raise the base a to obtain the number x

$$\log_a x = y \iff a^y = x$$

## Properties of logarithms

Be aware that logarithms are exponents, so the properties of logarithms stem from the [properties of powers]

Let a, p, q be positive real numbers with $a \neq 1$. Then

- **$\log_a a = 1$ $\quad$ $\log_a 1 = 0$ $\quad$ $\log_a a^x = x$ (for any x >0)**
- **Product Property**
  The logarithm of a product is the sum of the logarithms of each factor
  $$\log_a(p \cdot q) = \log_a p + \log_a q$$
- **Quotient Property**
  The logarithm of a quotient is the difference of the logarithms of the numerator and the denominator
  $$\log_a(p/q) = \log_a p - \log_a q$$
- **Power Property**
  The logarithm of a power is the product of the logarithm of the base and the exponent.
  $$\log_a p^q = q \cdot \log_a p$$
- **Change of base Formula**
  $$\log_a p = \frac{\log_b p}{\log_b a} \quad \text{with} \quad b > 0, b \neq 1$$

**Note:**
When using a calculator we can only evaluate logarithms with base e or 10.
The change of base formula allows us to express the logarithm of a number with any base in terms of a logarithm with base e or 10. So, we will be able to find any original logarithm by using the calculator.

**Example 7.**
Use $\log_3 2 \approx 0{,}63093$ and $\log_5 2 \approx 0{,}43068$ to evaluate the logarithms below:

| | |
|---|---|
| **$\log_3 162$** | |
| $= \log_3 (3^4 .2)$ | Factorise 162 |
| $= \log_3 3^4 + \log_3 2$ | Product Property |
| $= 4.\log_3 3 + \log_3 2$ | Power property |
| $= 4 + \log_3 2$ | $\log_a a = 1$ |
| $= 4 + 0{,}63093$<br>$= 4{,}63093$ | Replace $\log_3 2$ |

| | |
|---|---|
| **$\log_5 62{,}5$** | |
| $= \log_5 (625/10)$ | Replace 62,5 with 625/10 |
| $= \log_5 625 - \log_5 10$ | ………… Property |
| $= \log_5 5^4 - \log_5 (5.2)$ | Factorise 625 and 10 |
| $= 4.\log_5 5 - (\log_5 5 + \log_5 2)$ | ……………. property<br>and<br>………………..property |
| $= 4 - (1 + 0{,}43068)$ | $\log_a a = 1$<br>and<br>Replace $\log_5 2$ |
| $= 4 - 1{,}43068$<br>$= 2{,}56932$ | |

**Example 8.**
Check if the values of $\log_3 2$ and $\log_5 2$ given in the previous example are or not correct. Use the change of base formula and the calculator.

$$\log_3 2 = \frac{\log 2}{\log 3} \approx \frac{0{,}301029996}{0{,}477121255} \approx 0{,}630929754$$

$\log_5 2 = \text{———} \approx \text{——————} \approx \ldots\ldots\ldots\ldots\ldots\ldots$

**Example 9.**

Use the properties of logarithms to expand these expressions:

$\log_5 \frac{x^2 \cdot y^3 \cdot z^5}{t^2}$ and $\log_3 \frac{x^2 \cdot \sqrt[5]{y^4}}{\sqrt{z \cdot t^3}}$

$$\log_5 \frac{x^2 \cdot y^3 \cdot z^5}{t^2}$$

$$\log_5 x^2 \cdot y^3 \cdot z^5 - \log_5 t^2$$ Quotient property

$$\log_5 x^2 + \log_5 y^3 + \log_5 z^5 - \log_5 t^2$$ Product property

$$2\log_5 x + 3\log_5 y + 5\log_5 z - 2\log_5 t$$ Power property

$$\log_3 \frac{x^2 \cdot \sqrt[5]{y^4}}{\sqrt{z \cdot t^3}}$$

$$\log_3 \frac{x^2 \cdot y^{\frac{4}{5}}}{z^{\frac{1}{2}} \cdot t^{\frac{3}{2}}}$$ Write each root as a power

$$\log_3 x^2 \cdot y^{\frac{4}{5}} - \log_3 z^{\frac{1}{2}} \cdot t^{\frac{3}{2}}$$ Quotient property

$$\log_3 x^2 + \log_3 y^{\frac{4}{5}} - (\log_3 z^{\frac{1}{2}} + \log_3 t^{\frac{3}{2}})$$ Product property

$$2\log_3 x + \frac{4}{5} \cdot \log_3 y - \left(\frac{1}{2} \cdot \log_3 z + \frac{3}{2} \cdot \log_3 t\right)$$ Quotient property

$$2\log_3 x + \frac{4}{5} \cdot \log_3 y - \frac{1}{2} \cdot \log_3 z - \frac{3}{2} \cdot \log_3 t$$ Remove the bracket

**ACTIVITY 7.**

Use the change of base formula and the calculator to approximate the value of $\log_2 92000$, $\log_5 3$ and $\log_2 7$.Round to the nearest thousandth

**ACTIVITY 8.**

Use $\log_5 3$, $\log_5 2$,$\log_2 7$ to evaluate **$\log_2$ 3087** and **$\log_5 3{,}75$**

**ACTIVITY 9.**

Use the properties of logarithms to expand these expressions:

$\log_4 \frac{p^3 \cdot q^2 \cdot r}{s^5}$ and $\log_5 \frac{a^3 \cdot \sqrt[3]{b^3}}{\sqrt{c \cdot d^4}}$

A **logarithmic function** is an equation of the form

$f(x)=\log_a x$ (or $y=\log_a x$ )

where a is a positive real number ($a>0$ and $a\neq 1$)

**Properties**

- The graph of $y=\log_a x$ ($a>0$ and $a\neq 1$) never has a Y-intercept

  The X**-intercept** is always (1,0)

- If $a>1$,the function $y=\log_a x$ is **increasing**

  If $0<a<1$, the function $y=\log_a x$ is **decreasing**

- The **domain** of $f(x)=\log_a x$ ( $a>0$ y $a\neq 1$) is all positive real numbers

- The **range** of f is all real numbers

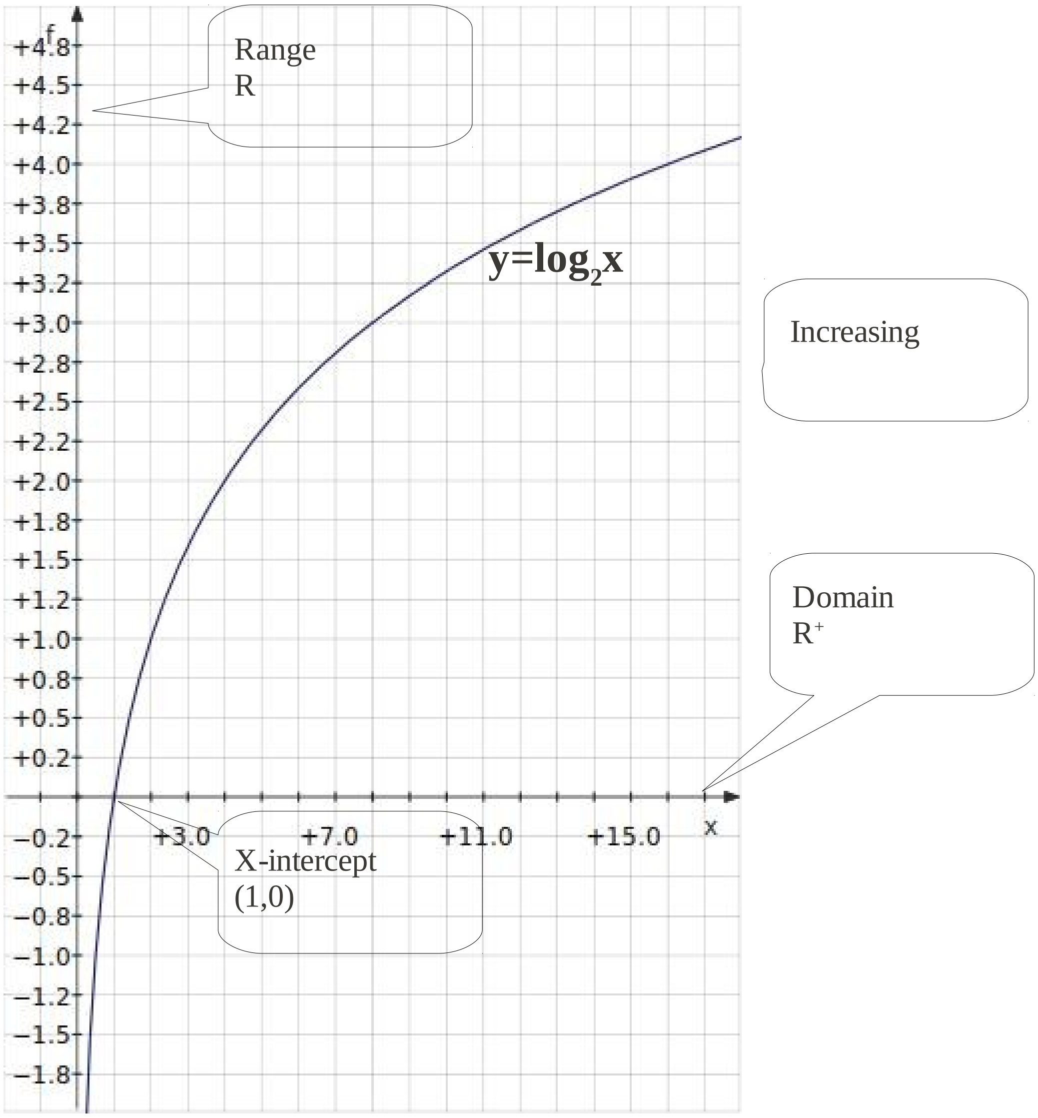
**Note:**
The logarithmic function is the inverse of the exponential function

Remember that $y=\log_a x \Longleftrightarrow a^y=x$

Compare the properties of both kinds of functions

**Example 10.**

Make a values table and graph the functions $y=\log_2 x$ .Then, sketch both functions $y=\log_2 x$ and $y=2^x$ on the same coordinate plane.



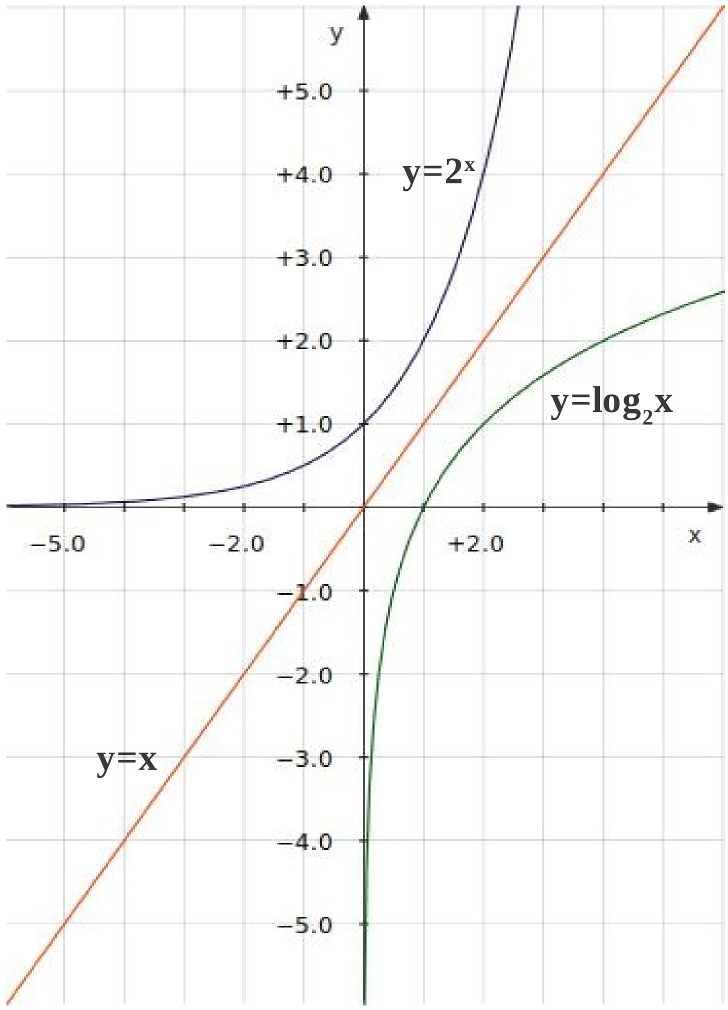

| x | y=log₂x |
|---|---|
| 0,25 | -2 |
| 0,5 | -1 |
| 1 | 0 |
| 2 | 1 |
| 4 | 2 |
| 8 | 3 |
| 16 | 4 |

$\log_2 0{,}25=y$
$2^y=0{,}25$
$2^y=1/4=2^{-2}$
$2^y=2^{-2}$
$y=-2$

$\log_2 0{,}5=y$
$2^y=0{,}5$
$2^y=1/2=2^{-1}$
$2^y=2^{-1}$
$y=-1$

$\log_2 8=y$
$2^y=8$
$2^y=2^3$
$y=3$

## $y=\log_2 x$ and $y=2^x$ on the same coordinate plane




Both graphs are reflection of each other about the line **y=x**

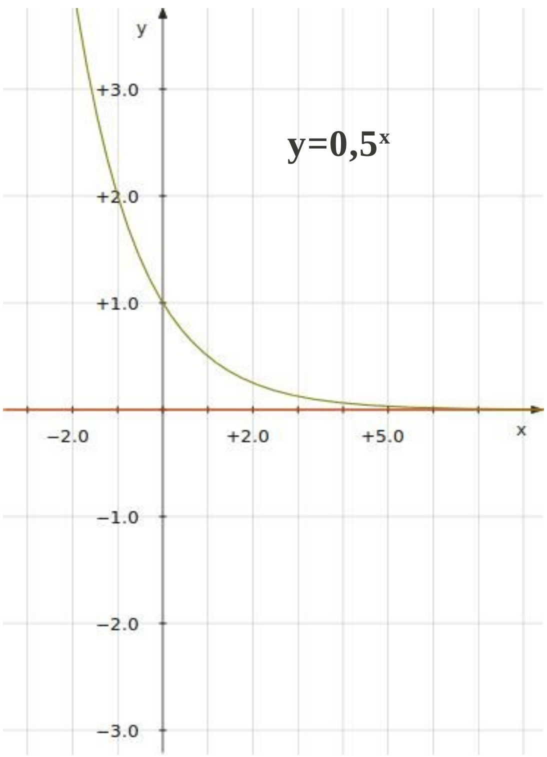
**Example 11.Graph the function**

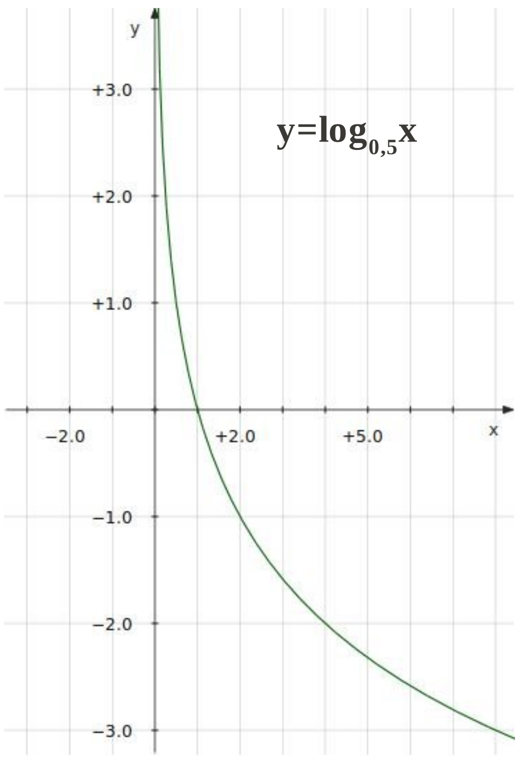
To graph the function $y=\log_{\left(\frac{1}{2}\right)} x$ we are going to use its inverse function, the exponential function y= $\left(\frac{1}{2}\right)^x$

| **x** | **y=** $\left(\frac{1}{2}\right)^x$ |
|---|---|
| -3 | 8 |
| -2 | 4 |
| -1 | 2 |
| 0 | 1 |
| 1 | 0.5 |
| 2 | 0.25 |
| 3 | 0,125 |

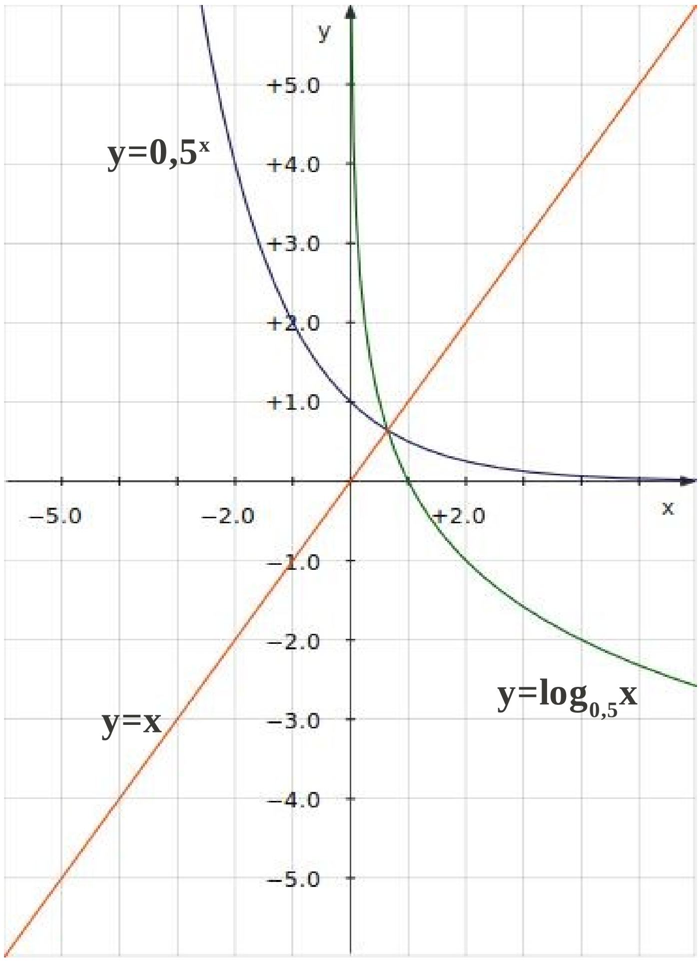
| **x** | $y=\log_{\left(\frac{1}{2}\right)} x$ |
|---|---|
| 8 | -3 |
| 4 | -2 |
| 2 | -1 |
| 1 | 0 |
| 0.5 | 1 |
| 0.25 | 2 |
| 0,125 | 3 |

Interchange the variables x and y



### $y=\log_{0,5}x$ and $y=0,5^x$ on the same coordinate plane




Both graphs are reflection of each other about the line **y=x**

**ACTIVITY 10.**

a) Make a values table of the function $y=\log_5 x$

| x | y=log₅x |
|---|---|
| 0,2 | |
| 1 | |
| 5 | |
| 25 | |
| 125 | |

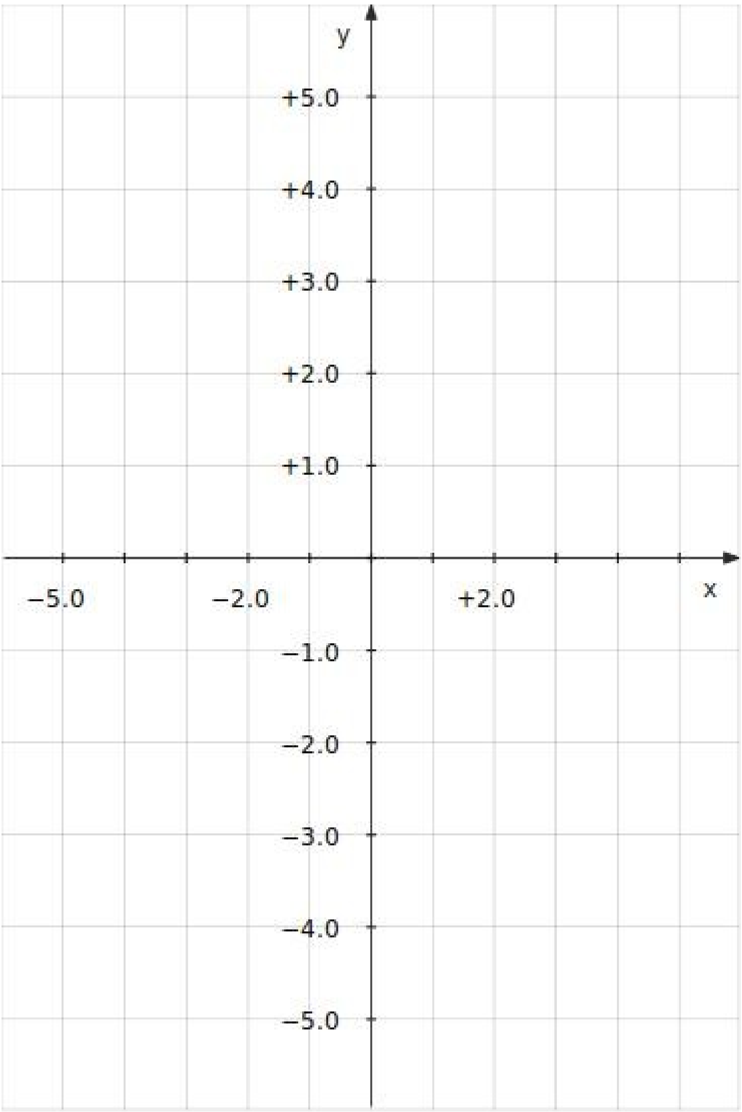
$\log_5 0,2=y$
$5^y=$......
$5^y=1/5=$....

$5^y=5^{\square}$
y=.......

$\log_5 25=y$
$5^y=$....

$5^y=5^{\square}$
y=

b) Plot some of the ordered pairs on the coordinate plane below and sketch the graph




c) In **ACTIVITY 1** you were asked to make a values table and graph the functions $y=5^x$ .Copy the table again and graph the function on the coordinate plane above. Compare both graphs

| **x** | **y=**$5^x$ |
|---|---|
| -1 | |
| 0 | |
| 1 | |
| 2 | |
| 3 | |

## ACTIVITY 11.

Complete the table:

| | $y=2^x$ | $y=\log_2 x$ | $y=(1/5)^x$ | $y=\log_{1/5} x$ |
|---|---|---|---|---|
| **Domain** | | | | |
| **Range** | | | | |
| **X-intercept** | | | | |
| **Y-intercept** | | | | |
| **Increasing or Decreasing** | | | | |

## ACTIVITY 12.

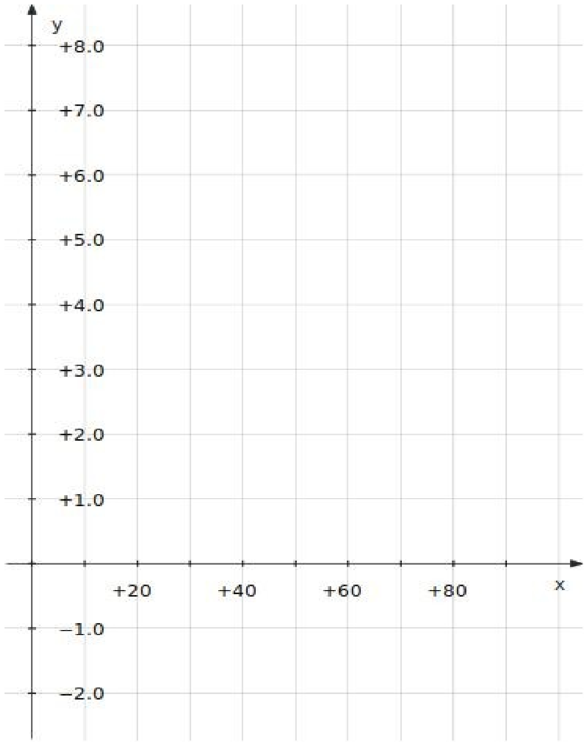
Graphing logarithmic functions. Video

Graphing logarithmic functions through the inverse. Video

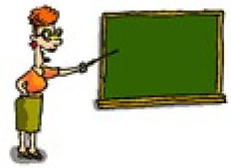
**ACTIVITY 13.**

Graph the logarithmic functions $y=\log x^3$, $y=\ln 3x$, $y=1+\log x$ using a spreadsheet to make the values table

| x | $y=\log x^3$ |
|---|---|
| 0,0001 | -12 |
| 0,001 | -9 |
| 0,01 | -6 |
| 0,1 | -3 |
| 1 | 0 |
| 5 | 2,0969100130 |
| 10 | 3 |
| 15 | 3,5282737772 |
| 20 | 3,9030899870 |
| 25 | 4,1938200260 |
| 30 | 4,4313637642 |
| 35 | 4,6322041331 |
| 40 | 4,8061799740 |
| 50 | 5,0969100130 |
| 60 | 5,3344537512 |
| 70 | 5,5352941200 |
| 80 | 5,7092699610 |
| 90 | 5,8627275283 |
| 100 | 6 |
| ....... | ,,,,,,,,,,, |




**Note :**
Remember that in y=logx the base is 10(common or decimal logarithm) and in y=lnx the base is e(natural or neperian logarithm).
Most science applications (Chemistry,Economy, Geography,Biology)
involve these kinds of logarithms



# Applications

- **Compound interest**

Interest is the amount of money that someone pays (or receives) when borrowing (or investing money).Compound interest is the interest that is computed on the initial amount deposited plus the interest earned .
Interest can be paid annually(once a year),quarterly (four times a year),monthly or daily.
The value C of a investment after t years is :

$$C=C_0\left(1+\frac{r}{k}\right)^{kt}$$

where $C_0$ is the original investment, r is the annual rate and k is the number of compounding periods

**Note:**
If interests are paid quarterly ,the number of periods(per year) is 4
If interests are paid monthly ,the number of periods is 12
If interests are paid daily ,the number of periods is 365

**Example 12.** If we invest 500 euros at 4,5%,compounded quarterly,what amount will result in 5 years?

$C_0=500$
r=4,5%=0,045
k=4
t=5

$$C=500\left(1+\frac{0{,}045}{4}\right)^{20}=500(1{,}01125)^{20}\approx 625{,}375$$

The value C of this investment after 5 years is approximately 625,375 Euros

**Example 13**. We invested 2000 euros in a plan 8 years ago. Now, that amount is worth 4045 euros. The interests were paid semiannually. Calculate the annual rate applied.

$C_0=2000$
r= unknown
k=2
t=8
C=4045

$$4045=2000\left(1+\frac{r}{2}\right)^{16}\rightarrow\left(1+\frac{r}{2}\right)^{16}=\frac{4045}{2000}=2{,}0225\rightarrow$$

$$1+\frac{r}{2}=\sqrt[16]{2{,}0225}\approx 1{,}045\rightarrow$$

$$r=(1{,}045-1).2=0{,}09$$

The annual interest rate applied is 9%

**Example 14**. If we deposit 3000 euros into an account at 5 % compounded annually .In how many years will the investment be worth 4432 euros

$C_0=3000$
r= 5%
k=1
t=unknown

$$4432=3000(1+0{,}05)^t$$

$$\frac{4432}{3000}=1{,}05^t$$

$$\log\left(\frac{4432}{3000}\right)=\log(1{,}05^t)$$

Take the logarithm of each side.You can choose either base e or base 10

$$\log\left(\frac{4432}{3000}\right)=t\cdot\log(1{,}05)$$

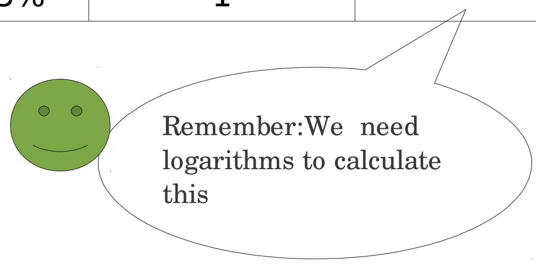
Use power property of logarithms to remove the variable from the exponent

$$t=\frac{\log\left(\frac{4432}{3000}\right)}{\log(1{,}05)}=\frac{0{,}169478497}{0{,}021189299}\approx 8$$

In 8 years approximately,the balance will be 4432 euros.

**ACTIVITY 14.** Complete the table :

| **Initial deposit** $C_0$ | **Future value** C | **Annual interest rate** r | **Number of compounding periods per year** n | **Time in years** t |
|---|---|---|---|---|
| 5000 | | 4,5% | 1 | 12 |
| | 22000 | 9% | 4 | 7 |
| 150000 | 196813 | | 2 | 5,5 |
| 20000 | 27258 | 3,5% | 1 | |



### ✔ Depreciation

It is very easy to deduce that if a product depreciates at a rate of r % each year and $V_0$ is the original value its value after a period of t years can be calculated using the formula:

$$V = V_0 \cdot \left(1 - \frac{r}{100}\right)^t$$

Similar to compound interest formula

**ACTIVITY 15.**
We bought a car valued at 12000 euros and we know that these cars depreciate at 18% each year.

a) What will the value of the car be after a year?

b) What will the value of the car be in 10 years' time?

c) How long will it take for the car to have a value of 6000 euros?

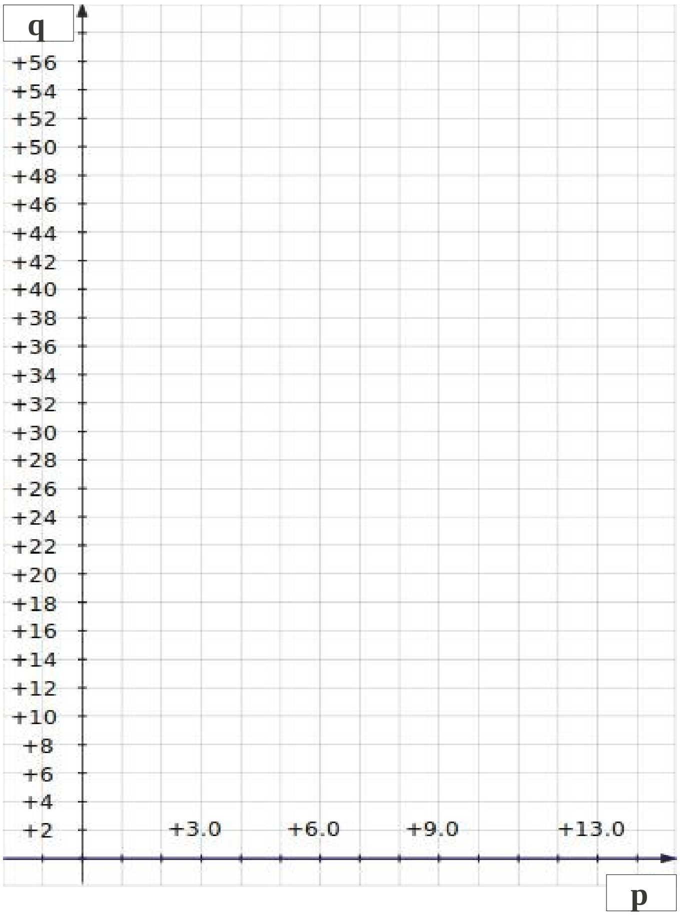
### ✔ Demand

Sometimes,supply,demand ,cost and revenue are modeled by quadratic functions but other times, they may be modeled by exponential functions

**Example 15.**
The demand function for certain product is $p=50.e^{-q/2}$ where p is the price per unit in Euros and q is the number of thousand of units demanded.

a) Make a table and graph the function

| **q** | **$p=50.e^{-q/2}$** |
|---|---|
| 2 | $50.e^{-1} \approx 18,4$ |
| 4 | $50.e^{-2} \approx$....... |
| 6 | ............... |
| 8 | ............... |
| 10 | ............... |




b) At what price per unit will the quantity demanded equal 20 units?.Use both the graph and the calculator and compare the results.

Using the graph ................................

Using the calculator: $p=50.e^{-20/2}=50.e^{-10}=$................

c) If the price per unit equals 10 euros ,how many units will be demanded?(round to the nearest thousand)

$$p=50\cdot e^{-q/2}$$

$$10=50\cdot e^{-q/2}$$

$$\text{———}=e^{-q/2}$$

$$0{,}2=e^{-q/2}$$

$$\ln(0{,}2)=\ln(.............)$$

Take the logarithm of each side. Here we use base e

ln e=........

$$.......=\frac{-q}{2}\cdot\ln e$$

Use the ............property of logarithms

$$q=.........$$

d) If the price increases,does the quantity demanded increase or decrease?

## ✔ Radioactivity

**Example 16**.

Carbon dating is a method used in archeology, geology, geophysics to estimate the age of organic material such as wood, charcoal, shell, and bone. This method uses the half-life of a radioactive isotope of carbon called carbon-14 to find the approximate age of an object less than 40,000 years old. Uranium-238 , which is an isotope with longer half-life, can be used to date older objects.

The amount of carbon-14 ,P,in an object after t years is given by:

$$\mathbf{P=P_0.(0{,}5)^{t/5570}}$$

where $P_0$ is the original amount of carbon-14 present in the object, t is the age of the object in years and 5570 is the approximate half-life of carbon-14.

a) A certain bone contains 150 mg of carbon-14, how much carbon-14 will it contain after a period of 250 years?

$$P=150.(0{,}5)^{t/5570}$$

t=250

$$P=150.(0{,}5)^{250/5570}$$

$$P=150.(0{,}5)^{\square}$$

P= ☐

b) The original amount of carbon-14 present in a bone was 125 mg and now it has 65 mg of this substance .What is the approximate age of the bone?

$$P = P_0 \cdot 0{,}5^{t/5570}$$

$$\ldots\ldots\ldots = \ldots\ldots\ldots \cdot 0{,}5^{t/5570}$$

$$\frac{\square}{\square} = 0{,}5^{t/5570}$$

$$\ldots\ldots\ldots = 0{,}5^{t/5570}$$

$$\ln(\ldots\ldots\ldots) = \ln(\ldots\ldots\ldots\ldots)$$

$$\ldots\ldots\ldots = \frac{t}{5570} \cdot \ln(0{,}5)$$

$$t = \ldots\ldots\ldots$$

Use the...................... property of logarithms

The bone is approximately ............years old.

**ACTIVITY 16.** Iodine-131, which has a half-life of eight days, is a radioactive isotope of the chemical element Iodine. It is used in medicine to monitor thyroid gland functioning, to treat thyroid cancer, and to locate tumours of the brain and of the liver.
Suppose that we give an injection that contains 6 micro grams of this substance to a patient.

a) How much Iodine is present after 8 days? (Pay attention:The half-life of Iodine-131 is 8 days)

b) How much Iodine is present after 11 days?

c) How many days will it take for the iodine level to reach 2 micro grams?

**ACTIVITY 17.**

**Radioactive Half Life**

## ✔ Population growth

**ACTIVITY 18.**

The table below shows the population( in thousands) of India from 1950 to 2010.
Suppose the function $\mathbf{P=P_0e^{kt}}$ models the growth of the population,where :
$\mathbf{P_0}$ is the original population
**P** is the population after a period of time,t
**k** is the rate of natural increase (growth rate)

| Year | Population |
|---|---|
| 1950 | 371 857 |
| 1955 | 406 661 |
| 1960 | 448 314 |
| 1965 | 496 934 |
| 1970 | 552 964 |
| 1975 | 617 432 |
| 1980 | 692 637 |
| 1985 | 774 775 |
| 1990 | 862 162 |
| 1995 | 953 148 |
| 2000 | 1 042 590 |
| 2005 | 1 130 618 |
| 2010 | 1 214 464 |

a) Use the data from 2005 to 2010 to find the value of k in the function $\mathbf{P=P_0e^{kt}}$

$$P=P_0\cdot e^{k\cdot t}$$

$$1214464=1130618\cdot e^{k\cdot 5}$$

$$\frac{1214464}{1130618}=e^{k\cdot 5}$$

$$1{,}074159442\approx e^{k\cdot 5}$$ Exponential form

$$k\cdot 5\approx \ln(1{,}074159442)$$ Logarithmic form

$$k\cdot 5\approx 0{,}071538441$$

$$k\approx 0{,}0143$$

b) Predict the population for the year 2015

t=5

$$P = 1214464 \cdot e^{0{,}0143 \cdot t}$$

$$P = 1214464 \cdot e^{0{,}0143 \cdot 5}$$

$$P \approx 1214464 \cdot e^{0{,}0715}$$

$$P \approx 1214464 \cdot 1{,}0741 \approx 1304456$$

Set the more recent data as the original population($P_0$=1214464)

c) What was the population of India in 2008 approximately?

t=3

$$P = \text{..........} \cdot e^{0{,}0143 \cdot t}$$

$$P = \text{............} \cdot e^{0{,}0143 \cdot \text{.....}}$$

$$P \approx \text{............} \cdot e^{\text{......}}$$

$$P \approx \text{............................}$$

Set $P_0$=1130618 as the original population

d) When will India's population be 2500 million?

2500 million=2500000 thousand

$$P = P_0 \cdot e^{k \cdot t}$$

$$2500000 = 1214464 \cdot e^{0{,}0143 \cdot t}$$

$$\frac{\boxed{\phantom{000000}}}{1214464} = e^{0{,}0143 \cdot t}$$

$$\text{................} \approx e^{0{,}0143 \cdot t}$$

Exponential form

$$\text{.............} \approx \ln(\text{................})$$

Logarithmic form

$$\text{................} \approx \text{................}$$

$$t \approx \text{...........}$$

## ✔ Seismology

**ACTIVITY 19.**

We use Richter scale to measure the intensity of an earthquake. The function that links the intensity I and the magnitude R on the Richter scale of an earthquake is:

$$R=\log(I/I_0)$$

where $I_0$ is a minimum intensity.

You will find more information and examples about the formula above in the page SOS Maths

a) Do you know the difference between intensity and magnitude of an earthquake?

You can visit the website British Geological Survey to answer the question.

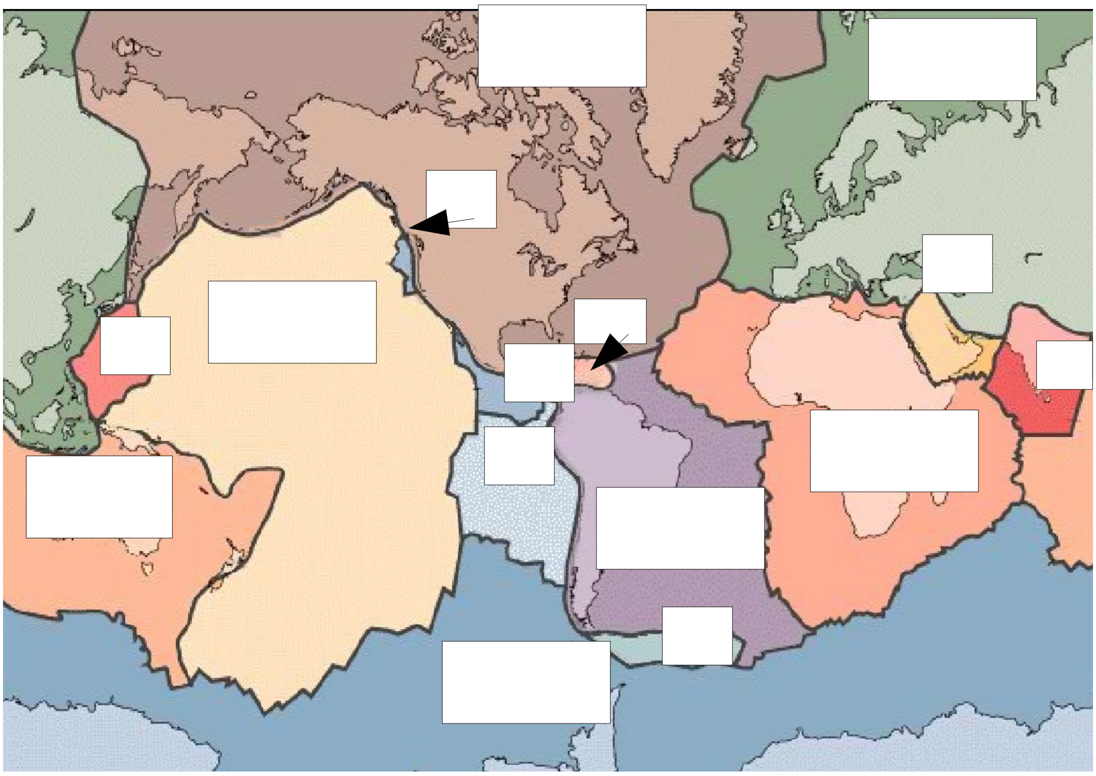
b) Could you tell us where the deadliest earthquake has struck over the past 110 years? Could you state the name,location and magnitude of the five most devastating ?

The link US geological Survey can help you to find the answer.

c) As you will probably remember,the Earth's lithosphere is broken into oceanic and continental plates which can slide over the asthenosphere. These plates are in constant motion and cause,when interacting, important geological processes such as the formation of mountain belts, earthquakes, and volcanoes.

- To know different types of collisions click here: ThinkQuest
- Fill in the blanks with the correct name of the plate

| |
|---|
| Eurasian Plate, Australian Plate, Philippine Plate, North American Plate<br>Pacific Plate, Eurasian Plate, India Plate, Australian Plate, Saudi Plate,<br>African Plate, South American Plate, Antarctic Plate, Nazca Plate,<br>Caribbean Plate, Cocos Plate, Scotia Plate,Juan de Fuca Plate |



Source:http://www.iris.edu:

d) The 2010 Haiti and the 1976 China earthquakes measured 7.0 and 7.5 on the Richter scale respectively. Find the intensity of each earthquake.

**1976 China earthquake**

$$7{,}5=\log(I/I_0)$$

$$10^{7{,}5}=I/I_0$$ Exponential form

$$31622776{,}602\approx I/I_0$$ Using the calculator

$$I\approx 31622776{,}602 . I_0$$ Multiplying by $I_0$

The intensity is 31622776,602 times $I_0$

## 2010 Haiti earthquake

$7 = \log(I/I_0)$

[ ] — Exponential form

[ ] — Using the calculator

$I \approx$ [ ] $.I_0$ — Multiplying by $I_0$

The intensity is [ ] times $I_0$

e) How many times more severe was the China earthquake than the Haiti one?

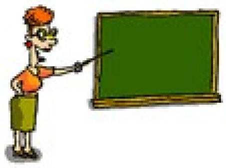
f) Use a spreadsheet to calculate the intensities of the earthquakes recorded over the past 20 years. Compare these intensities.

| Year | Magnitude R | Intensity($I=10^R .I_0$) |
|---|---|---|
| 1990-Iran | 7,4 | 25118864,3150958 $I_0$ |
| 1990-Philippine Islands | 7,7 | 50118723,3627272 $I_0$ |
| 1991-India | 7 | 10000000 $I_0$ |
| 1992-Indonesia | 7,5 | 31622776,6016838 $I_0$ |
| 1993-India | ....... | |
| ....... | ....... | |
| ....... | ....... | |
| 2010-Haiti | 7 | |

| | | |
|---|---|---|
| The 1992 earthquake was | 3,162 | more intense than the 1991 one |
| The 2010 earthquake was | ? | more intense than the 1991 one |
| ............ | ....... | .................................. |

g) Did you find words whose meaning you don't know in the texts? Make your own table where you must write down the definition, a simple sentence containing the word and the pronunciation .

| Word | Definition | Sentence | Pronunciation |
|---|---|---|---|
| lithosphere | The crust and upper mantle of the Earth | The **lithosphere** is the solid shell of our planet | /ˈlɪθəˌsfɪər/ |
| asthenosphere | ….......................... | ….......................... | ….......................... |
| ….......................... | ….......................... | ….......................... | ….......................... |



# Revision and Assessment

**ACTIVITY 20.**Revise your vocabulary
Choose a word in the box and fill the blanks below:

| |
|---|
| **Quotient Property ,exponential function,Change of base formula, decreasing , logarithmic function, Product Property , increasing,** |

- An ________ is an equation of the form $f(x)=a^x$ ( a>0 , a≠1)
- A ________ is an equation of the form $f(x)=\log_a x$ (a>0, a≠1 )
- If a>1,the function $y=a^x$ is ________ (exponential growth).
- If 0<a<1, the function $y=a^x$ is ________ (exponential decay).
- ________ :The logarithm of a quotient is the difference of the logarithms of the numerator and the denominator.
- ________ :The logarithm of a product is the sum of the logarithm of each factor.
- ________

$$\log_a p=\frac{\log_b p}{\log_b a} \quad with \ \ b>0, b\neq 1$$

**ACTIVITY 21.**Complete the table to revise the keywords' unit

| **Terms or expressions** | **Definition** | **Example** | **Pronunciation** |
|---|---|---|---|
| Exponential functions | | | |
| Logarithmic functions | | | |
| Logarithms | | | |
| Domain | | | |
| Range | | | |
| X-intercept | | | |
| Y-intercept | | | |
| Translations | | | |
| Stretching | | | |
| Reflection | | | |
| Inverse | | | |
| Growth | | | |
| Decay | | | |
| …………………… | | | |

**ACTIVITY 22.**Evaluate your learning through the following tests and on line resources.

| | |
|---|---|
| Exponential and logarithmic properties | ■ Tests<br>LOGARITHMIC EQUATIONS<br>EXPONENTIAL EQUATIONS<br>■ Test<br>LOGARITHMS AND EXPONENTIALS |
| Graphing Exponential and Logarithmic functions | ■ Test<br>EXPONENTIAL DECAY |
| Logarithms and exponentials as inverses | ■ Videos<br>LOGARITHMS AS INVERSES |
| Applications | ■ Worksheet<br>EXPONENTIAL POPULATION GROWTH |

**ACTIVITY 23.** Revise the properties of powers

- **Product of Powers**

  To multiply two powers that have the same base, we ............the exponents.

$$a^m \cdot a^n = a^{m+n}$$

- **Quotient of Powers**

  To divide two powers that have the same base,we .............. the exponents.

$$a^m : a^n = a^{m-n}$$

- **Power of a Power**

  To find the power of a power,we ........................ the exponents.

$$(a^m)^n = a^{m.n}$$

- **Power of a Product**

  To find the power of a product, we do the power of each factor and then, we --------------------.

$$(a \cdot b)^n = a^n . b^n$$

- **Power of a Quotient**

  To find the power of a quotient, we do the power of the ..............................and the power of the ...............................

$$(a : b)^n = a^n : b^n$$

- **Zero Exponent**

  Any non zero number raised to the zero power is ............

$$a^0 = 1$$